Problem 1.1.20: Use Theorem 1.1.6 to prove Corollary 1.1.7

Corollary 1.1.7

A sequence $(d_1, d_2, \ldots, d_n)$ of nonnegative integers such that $d_1 >= d_2 >= \ldots >= d_n$ is graphic if and only if the sequence $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic.

Proof.

Call $(d_1, d_2, \ldots, d_n)$ the *original* sequence and $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n)$ the *derived* sequence.

We must show both that:

- The original sequence is graphic if the derived sequence is graphic.
- The derived sequence if graphic if the original sequence is graphic.

We do this in two steps.

Part 1: The original sequence is graphic if the derived sequence is graphic.

Suppose we have a set of integers $d_1, d_2, \ldots, d_n$ and consider the sequence $S_1 = (s_1 = d_2 - 1, s_2 = d_3 - 1, \ldots, s_{d_1} = d_{d_1+1} - 1, s_{d_1+1} = d_{d_1+2}, s_{d_1+2} = d_{d_1+3}, \ldots, s_{n-1} = d_n)$. We will show that if S_1 is graphical, the sequence $S_2 = (d_1, d_2, \ldots, d_n)$ is, too.

To see this, assume that S_1 is graphic. Then, by the definition of a graphic sequence, there exists a graph G_1 which contains nodes of degree $(s_1, s_2, \ldots, s_n)$. We can label these nodes $v_2, v_3, \ldots, v_n$ in such a way that the node with label v_i has degree s_{i-1}.

Now, consider the graph formed by adding a new node v_1 and connecting it to the first d_1 nodes. The degrees of nodes v_{d_1+1} through v_n are unchanged, because we haven't added an edge to them. However, the degrees of the first d_1 nodes have increased by one. The degree of the new node, v_1 is d_1. Therefore, the overall degree sequence is now $(d_1, s_1 + 1 = d_2 - 1 + 1 = d_2, s_2 + 1 = d_3 - 1 + 1 = d_3, \ldots, s_{d_1} + 1 = d_{d_1+1} - 1 + 1 = d_{d_1}, s_{d_1+2} = d_{d_1+1}, \ldots, s_{n-1} = d_n$. In other words, the new graph is the graph with degree sequence S_2. Therefore, S_2 is graphic.

Thus, if the derived sequence is graphic, the original sequence must be graphic too.

Part 2: The derived sequence is graphic if the original sequence is graphic.

Let $S_2 = (d_1, d_2, \ldots, d_n)$ for some integers $d_1, d_2, \ldots, d_n$. Since the labels (d_1 and so forth) are arbitrary, we can assume, without loss of generality, that they are in non-increasing order such that $d_1 >= d_2 >= \ldots >= d_n$.

We will demonstrate that the sequence $S_1 = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n)$ is graphic.

By theorem 1.1.6, there is a graph G_2 which contains nodes $v_1, v_2, \ldots, v_n$ with degrees $d_1, d_2, \ldots, d_n$, respectively, such that v_1 is adjacent to $v_2, v_3, \ldots, v_{d_1+1}$.

Consider the graph G_3 formed by removing v_1 and all its incident edges. The degree sequence of G_3 is a sequence of $n - 1$ integers representing the degrees of vertices $v_2, v_3, \ldots, v_n$. Since v_1 was adjacent to the first d_1 nodes, each of these now has degree $d_i - 1$ for i from 2 to $d_i + 1$. The remaining $n - d_1$ nodes have the degrees they had before: $d_{d_1+2}, d_{d_1+3}, \ldots, d_n$. Therefore the degree sequence of this graph is $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$. We have constructed a simple graph with degree sequence S_1. By the definition of graphic, S_1 is graphic.

Summary

Since both parts are true, the original sequence if graphic if and only if the derived sequence is also graphic.

Q.E.D.